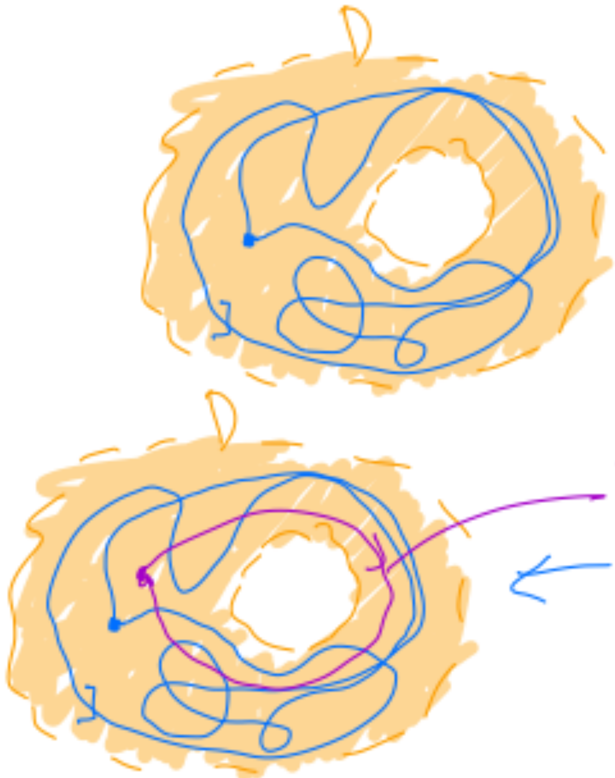
Comment's on Cauchy's Theorem (Cauchy–Gorsaut Thm).

Hypotheses:

f is a holom. fcn on a domain D [open, connected set in $\mathbb{C}$]. If α is a continuous, rectifiable, closed curve that is homotopic to a point in D, then
[can be deformed within D to a pt.]

$$\int_{\alpha} f(z)\,dz = 0.$$



This α is also homotopic to a pt in D

This β is <u>not</u> homotopic to a pt in D

More stuff about holomorphic fcns.

$$f : \underset{\text{open}}{D} \to \mathbb{C} \quad \text{is} \quad \text{holom}$$

on the domain D

$\iff$ f is $\mathbb{C}$-diff'ble on D.

$\iff$ $\forall z_0 \in D, \exists \varepsilon > 0$ s.t. $B(z_0, \varepsilon) \subseteq D$ and f is $\mathbb{C}$-diff'ble on $B(z_0, \varepsilon)$.

literal defn.

f is $\mathbb{C}$-diff. at $a \in D$

$$\iff \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \text{ exists}$$

$$\iff \lim_{z \to a} \frac{f(z) - f(a)}{z-a} = f'(a) \text{ exists}$$

$$\iff \lim_{z \to a} \frac{f(z) - f(a) - f'(a)(z-a)}{z-a} = 0$$

$$\iff f(z) - f(a) - f'(a)(z-a) = o(z-a)$$

"little o of $(z-a)$"

$g(z) = o(\text{bubba}) \iff \left|\frac{g(z)}{\text{bubba}}\right| \to 0$ as bubba $\to 0$

$G(z) = O(\text{bubba}) \iff \left|\frac{G(z)}{\text{bubba}}\right| \leq C$ for some constant C.

"big O of bubba"

f is $\mathbb{C}$-diff'ble at $a \in D$

$$\iff \underline{f(z) = f(a) + f'(a)(z-a) + o(z-a)}.$$

↳ $f(z)$ is close to its complex linear approx. (1st degree Taylor poly)

ie $f(z) \approx f(a) + f'(a)(z-a)$

Thm (Cauchy Integral Formula – CIF)

Let f be holomorphic on a domain $D \subseteq \mathbb{C}$, and let $z \in D$. Then

$$f(z) = \frac{1}{2\pi i}\int_\alpha \underbrace{\frac{f(\omega)}{\omega - z}}_{\text{holom on } D \setminus \{z\}} d\omega,$$

where α is any curve homotopic to a tiny circle oriented CCW in the domain around z & homotopic to z.

(Often stated with

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(z,\varepsilon)} \frac{f(\omega)}{\omega - z} d\omega$$

).

Proof:

f is C diff'ble on D

$$\int_{\alpha} \frac{f(\omega)}{\omega - z}\, d\omega \overset{\downarrow}{=} \int_{\alpha} \frac{f(\omega) - f(z) - f'(z)(\omega - z)}{\omega - z}\, d\omega$$

$$+ \int_{\alpha} \frac{f(z)}{\omega - z}\, d\omega + \int_{\alpha} f'(z)\, d\omega$$

$$= \int_{\alpha} \frac{f(\omega) - f(z) - f'(z)(\omega - z)}{\omega - z}\, d\omega + f(z) \underbrace{\int_{\alpha} \frac{1}{(\omega - z)}\, d\omega}_{2\pi i} + f'(z) \underbrace{\int_{\alpha} 1\, d\omega}_{0}$$

$2\pi i$ (by Fundamental Integral Calculation)

0 Cauchy's Thm

$$\underset{\text{Defom Thm}}{=} \int_{\partial B(z,\varepsilon)} \frac{o(\omega - z)}{\omega - z}\, d\omega + 2\pi i\, f(z)$$

$f(\omega) = f(z) + f'(z)(\omega - z) + o(\omega - z)$

Let $\varepsilon \to 0^+$

$$\left| \int_{\partial B(z,\varepsilon)} \frac{o(\omega - z)}{\omega - z}\, d\omega \right| \le \int_{\partial B(z,\varepsilon)} \left| \frac{o(\omega - z)}{\omega - z} \right| |d\omega|$$

$\omega = \varepsilon e^{i\theta} + z$ $(-\pi \le \theta \le \pi)$

$$= \int_{\theta=-\pi}^{\pi} \left| \frac{o(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \right| \left| i \varepsilon e^{i\theta} \, d\theta \right|$$

$$= \int_{\theta=-\pi}^{\pi} \frac{|o(\varepsilon e^{i\theta})|}{\cancel{\varepsilon}} \cdot \cancel{\varepsilon} \, |d\theta|$$

ML ineq

$$\leq \max_{\theta} |o(\varepsilon e^{i\theta})| \, 2\pi \to 0$$

as $\varepsilon \to 0$,

$$\frac{o(\varepsilon e^{i\theta})}{|\varepsilon e^{i\theta}|} \to 0.$$

We had

$$\int_{\alpha} \frac{f(w)}{w-z} \, dw = \underbrace{\int_{\alpha} \frac{o(w-z)}{w-z} \, dw}_{\to 0} + 2\pi i f(z)$$

0 as $\varepsilon \to 0$,

but it's a constant

$\therefore = 0$.

$$\Rightarrow \int_{\alpha} \frac{f(w)}{w-z} \, dw = 2\pi i f(z).$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(w)}{w-z} \, dw}$$

Consequences:

① Pick $\alpha = z + \varepsilon e^{it} \quad -\pi \le t \le \pi$

for small enough $\varepsilon > 0$ s.t. the α is inside the domain of f as well as $B(z_0, \varepsilon)$.

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{\alpha} \frac{f(\omega)}{\omega - z} \, d\omega$$

$\omega = \alpha(t)$

$d\omega = \alpha'(t) dt$

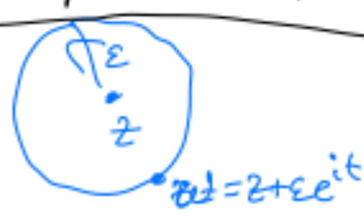
$= i\varepsilon e^{it}$

$$\Rightarrow f(z) = \frac{1}{2\pi \cancel{i}} \int_{-\pi}^{\pi} \frac{f(z + \varepsilon e^{it})}{\cancel{\varepsilon e^{it}}} \cdot \cancel{i \varepsilon e^{it}} \, dt$$

$$\Rightarrow f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z + \varepsilon e^{it}) \, dt$$

$\boxed{f(z) = \text{average value of } f \text{ on the circle of radius } \varepsilon \text{ around } z.}$

$\boxed{\text{Mean value Property of Holomorphic functions}}$



$\Rightarrow$ This implies that the real & imaginary parts of $f(z) = u(x,y) + i(v(x,y))$ also have this property. (called the <u>mean value property of harmonic functions</u>).

<u>Note</u>: If $u(x,y)$ is <u>any</u> harmonic fcn on a domain D in $\mathbb{R}^2$, $\exists\, v(x,y)$ s.t. $u + iv$ is holomorphic.

(Why? If we know u, we know u_x & u_y

$\Rightarrow$ Solve for $v(x,y) = -\int_0^x u_y(t,y)\,dt + C(y)$.

[Since we want $v_x = -u_y$]

$v(x,y) = \int_0^y u_x(x,t)\,dt + C_2(x)$

[Since want $u_x = v_y$].

In fact v is unique up to a constant.)

v is called a harmonic conjugate of u.

Let's check with an example:

$$f(z) = z^2.$$

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(0+\varepsilon e^{it})\,dt$$

LHS $= f(0) = 0$

RHS $= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varepsilon e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varepsilon^2 e^{2it}\,dt$

$$= \frac{\varepsilon^2}{2\pi}\,\frac{e^{2it}}{2i}\Bigg|_{-\pi}^{\pi} = \frac{\varepsilon^2}{2\pi}\left(\frac{1}{2i}-\frac{1}{2i}\right)$$

$$= 0 \checkmark$$

CIF: $f(z) = z^2 = \text{LHS}$ $\quad \alpha(t) = z + e^{it}$ $\quad -\pi \le t \le \pi$

$\varepsilon = 1$

$$\text{RHS} = \frac{1}{2\pi i}\int_\alpha \frac{\omega^2}{\omega - z}\,d\omega$$

$\omega = \alpha(t) = z + e^{it}$

$d\omega = ie^{it}\,dt$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{(z+e^{it})^2}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(z^2 + 2e^{it} z + e^{2it} \right) dt$$

$$= \frac{1}{2\pi} \left(z^2 t + \frac{2e^{it}}{i} z + \frac{e^{2it}}{2i} \right) \Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \Bigg(z^2 \pi + \cancel{\frac{2(-1)}{i}} z + \cancel{\frac{1}{2i}}$$

$$- \left[z^2(-\pi) + \cancel{\frac{2(-1)}{i}} z + \cancel{\frac{1}{2i}} \right] \Bigg)$$

$$= \frac{1}{2\pi} \left(2\pi z^2 \right) = z^2. \checkmark$$